

Thermodynamic analysis of Collins helium liquefaction cycle

M.D. Atrey

Cryogenics Section, Centre for Advanced Technology, Indore 452 013, India

Received 13 March 1998

The present paper gives a thermodynamic analysis of the Collins helium liquefaction cycle with two reciprocating expanders. The results of the analysis make it clear that, for a given efficiency of expanders and effectiveness of heat exchangers, there exists an optimum mass flow fraction of total helium gas mass flow rate that should be diverted through the expanders for which liquid yield is maximum and net power input is minimum. The analysis quantitatively studies the effect of expander efficiency and heat exchanger effectiveness on the performance of the liquefier. It gives final steady state temperature distribution across the cycle, which is essential data for carrying out the preliminary design of various components in the cycle. © 1999 Elsevier Science Ltd. All rights reserved

Keywords: helium liquefier; thermodynamic analysis

The helium liquefier based on the Collins cycle normally consists of six heat exchangers and two reciprocating expanders. The design of these would be possible only when the design data in terms of nodal temperatures across heat exchangers and expanders, effectiveness of heat exchangers and efficiencies of expanders, mass flow rate through compressor, expanders and J-T valve, etc., are made available. The design is quite critical at low temperatures due to changes in thermophysical properties of helium gas. Different parameters like heat exchanger effectiveness (ϵ), expander efficiencies (η_1 and η_2), temperatures of gas before expansion, total mass flow rate (\dot{m}), mass flow fraction through expanders ($\dot{m}_{e1} + \dot{m}_{e2}$) etc., affect the performance of the liquefier. Quite a bit of simulation work has been presented in the earlier developmental period of these machines. Hubbell and Toscano¹ presented an entropy generation concept for carrying out thermodynamic optimisation of the helium liquefaction cycle. Minta and Smith² used a similar method of minimisation of the generated entropy in a cycle model with continuous precooling. Khalil and McIntosh³ carried out an exhaustive study to optimise inlet pressure, temperature of first expander and number of expanders. Also, Hilal⁴ analysed the effect of the number of expansion engines in cascade form or in the independent form and pressure on the COP of the refrigerator and liquefier. He showed that there is a significant increase in coefficient of performance (COP) value in case of independent expansion engines over the one obtained in case of cascaded form. The required optimum pressure is also lower. In the recent past, this topic of cycle simulation is again gaining importance due to the increasing need of the efficient helium liquefiers for cooling of superconducting magnets. Nobutoki et al.⁵ and Malaen et al.⁶ have

presented simulation programs for the Large Helical Device (LHD) and the Large Hadron Collider (LHC) projects, respectively, for helium liquefaction/refrigeration plants in order to estimate, understand and analyse the performance of cryogenic processes before investing in the actual manufacturing of these plants. However, none of these analyses have referred to the optimum fraction of total mass flow rate that has to be diverted through the expanders, and have also not quantitatively analysed the effect of expander efficiency and heat exchanger effectiveness on this fraction and finally on the performance of the liquefier. This may be due to the fact that many of these simulation programs are classified in nature. The cold produced in the expanders is directly proportional to the mass flow rate diverted through them and the liquefaction yield is proportional to the remaining mass flow rate that passes through the J-T valve. If the total mass flow rate that goes through the first and the second expander, ($\dot{m}_{e1} + \dot{m}_{e2}$), is less than a minimum required quantity, there would not be any liquefaction of helium gas. This is due to the fact that the gas would never attain a low enough temperature for liquefaction due to insufficient refrigeration effect, and instead the machine would act as a refrigerator. Also, the parameters like heat exchanger effectiveness and expander efficiency affect the liquefaction yield considerably. The inlet temperature of the gas at the expander depends on the heat exchanger effectiveness at every stage and also on the mass flow rates through different parts of the cycle. The present paper aims to carry out an exhaustive simulation study of the Collins helium liquefier with two reciprocating engines. The analysis can also be extended or interpreted for cycles with turboexpanders.

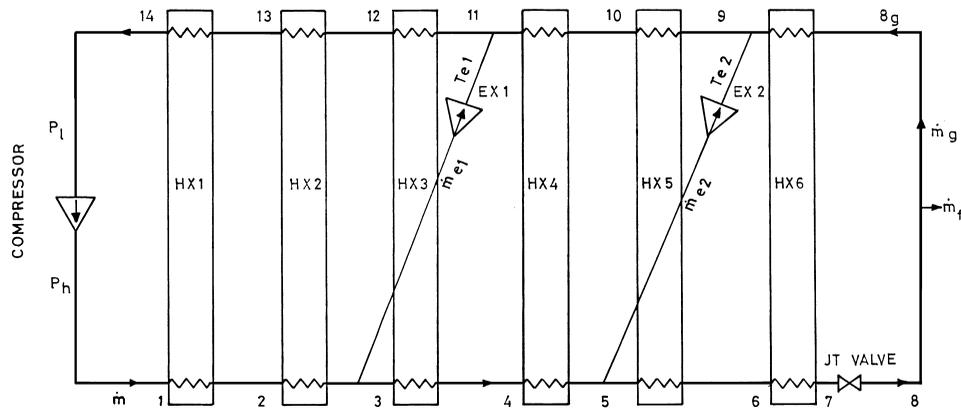


Figure 1 Schematic of the Collins helium liquefaction cycle

Thermodynamic analysis

Collins cycle

The Collins cycle or the modified Claude cycle is the one which is normally used for helium liquefaction. Figure 1 gives a schematic diagram of the Collins cycle and Figure 2 gives its process representation on the T-S diagram. Six heat exchangers, identified as HX1, HX2... HX6, respectively, and two reciprocating expanders identified as EX1 and EX2 are shown in the schematic. \dot{m} is the total mass flow rate of the helium gas through the compressor while \dot{m}_{e1} and \dot{m}_{e2} are the mass flow rates diverted through the expansion engine number 1 and 2, respectively. \dot{m}_f is the liquefaction yield. The present thermodynamic analysis is based on the steady state conditions at the time of liquefaction. ϵ_1 to ϵ_6 represent the effectiveness of the heat exchangers from HX1 to HX6, respectively, and η_1 and η_2 represent the isentropic efficiencies of the expanders 1 and 2, respectively, P_h and P_l represent discharge and suction pressure of the compressor. The heat exchangers, HX1 and

HX2, shown in Figure 1, can be integrated together to reduce number of variables in the analysis. However, this has been kept separate in the present analysis to study the option of LN2 precooling for the warm heat exchanger up to a desired temperature level. This calls for special attention to attribute effectiveness to each division of the warm heat exchanger. It should be noted that if each of the two warm heat exchangers has 96% effectiveness, the integrated heat exchanger then would have a higher resultant effectiveness than 96%.

Assumptions

Following assumptions are made for carrying out the analysis.

1. The maximum pressure (P_h) in the system is 15 bar and the minimum pressure (P_l) is 1 bar.
2. The temperature of the gas after compression is 300 K and the return stream temperature of the helium gas after liquefaction is at its boiling point, i.e. 4.21 K.
3. The pressure drop in the heat exchangers is negligible.
4. The J-T expansion is a perfect isenthalpic process.
5. Heat in-leak in the system is negligible.
6. Effectiveness of heat exchangers and efficiencies of expanders are assumed to be constant; their dependence on pressure, temperature and mass flow rate is ignored.

Analysis

The thermophysical properties of the helium gas, at different temperatures and pressures, are taken from Van Sciver⁷. For any intermediate temperatures, the values for enthalpy, entropy, etc. are linearly interpolated. Applying the first law of thermodynamics to the system, excepting the compressor, for the steady state condition, the ratio of liquid yield to the total mass flow rate, y , is given as follows:

$$y = \frac{\dot{m}_f}{\dot{m}} = \frac{h_{14} - h_1}{h_{14} - h_f} + x_1 \frac{\Delta h_{e1}}{h_{14} - h_f} + x_2 \frac{\Delta h_{e2}}{h_{14} - h_f} \quad (1)$$

where $x_1 = \dot{m}_{e1}/\dot{m}$ and $x_2 = \dot{m}_{e2}/\dot{m}$ and Δh_{e1} and Δh_{e2} are the net enthalpy changes in helium occurring in expander number 1 and 2, respectively. h represents enthalpy at the respective points.

A computer program is developed to analyse thermal

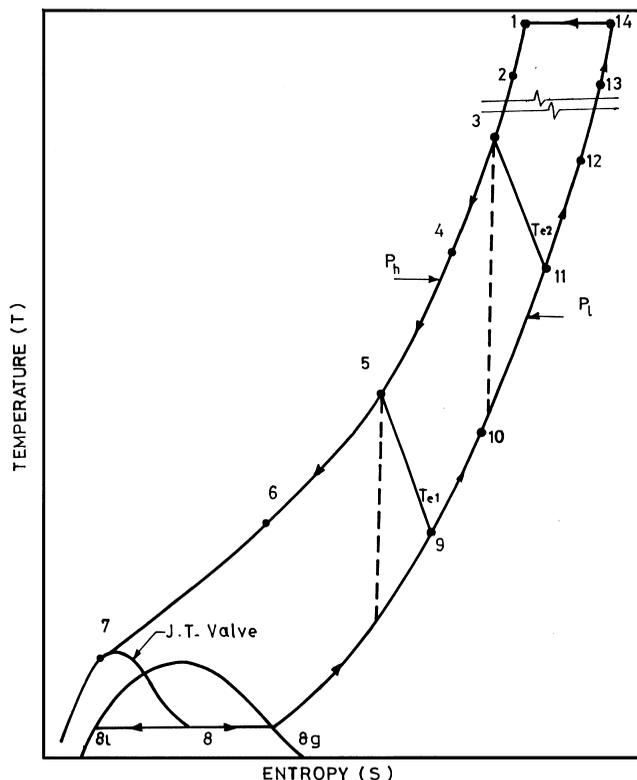


Figure 2 T-S diagram of the Collins helium liquefaction cycle

performance of the combined unit of six heat exchangers and two expanders along with the J-T expansion valve. A detailed flow chart for this analysis is given in Figure 3. The crucial part of the analysis is that only two temperatures are known initially, that is, the temperature of the gas after compression, T1, equal to 300 K, and the return stream temperature of the gas after liquefaction, T8g, equal to 4.21 K. All the intermediate temperatures are unknown variables excepting the effectiveness of all the heat exchangers and the efficiencies of the expanders. The effectiveness of heat exchangers, ϵ , is defined as:

$$\epsilon = \text{actual heat transfer}/\text{maximum possible heat transfer}$$

$$\epsilon = C_c(T_{co} - T_{ci})/C_{\min}(T_{hi} - T_{ci}) \quad (2)$$

$$= C_h(T_{hi} - T_{ho})/C_{\min}(T_{hi} - T_{ci}) \quad (3)$$

where, C is capacity rate, product of mass flow rate and

specific heat capacity of gas. Suffix c and h represent cold and hot fluid respectively, C_{min} indicates smaller quantity of C_c and C_h, suffix o and i represent outlet and inlet, respectively.

The efficiency of an expander, η , is defined as:

$$\eta = \text{actual enthalpy drop}/\text{maximum possible enthalpy drop} = (h_1 - h_2)/(h_1 - h_{2i}) \quad (4)$$

where h₁ is the enthalpy at the point from where expansion takes place, h₂ is the enthalpy at the actual point after expansion, h_{2i} is the enthalpy at the point if the expansion is isentropic in nature.

Based on the enthalpy balances in the system and incorporating ϵ and η definitions at respective nodal points, the temperatures at different nodes are calculated in an iterative manner. Appendix A gives all the equations for different important nodes in detail.

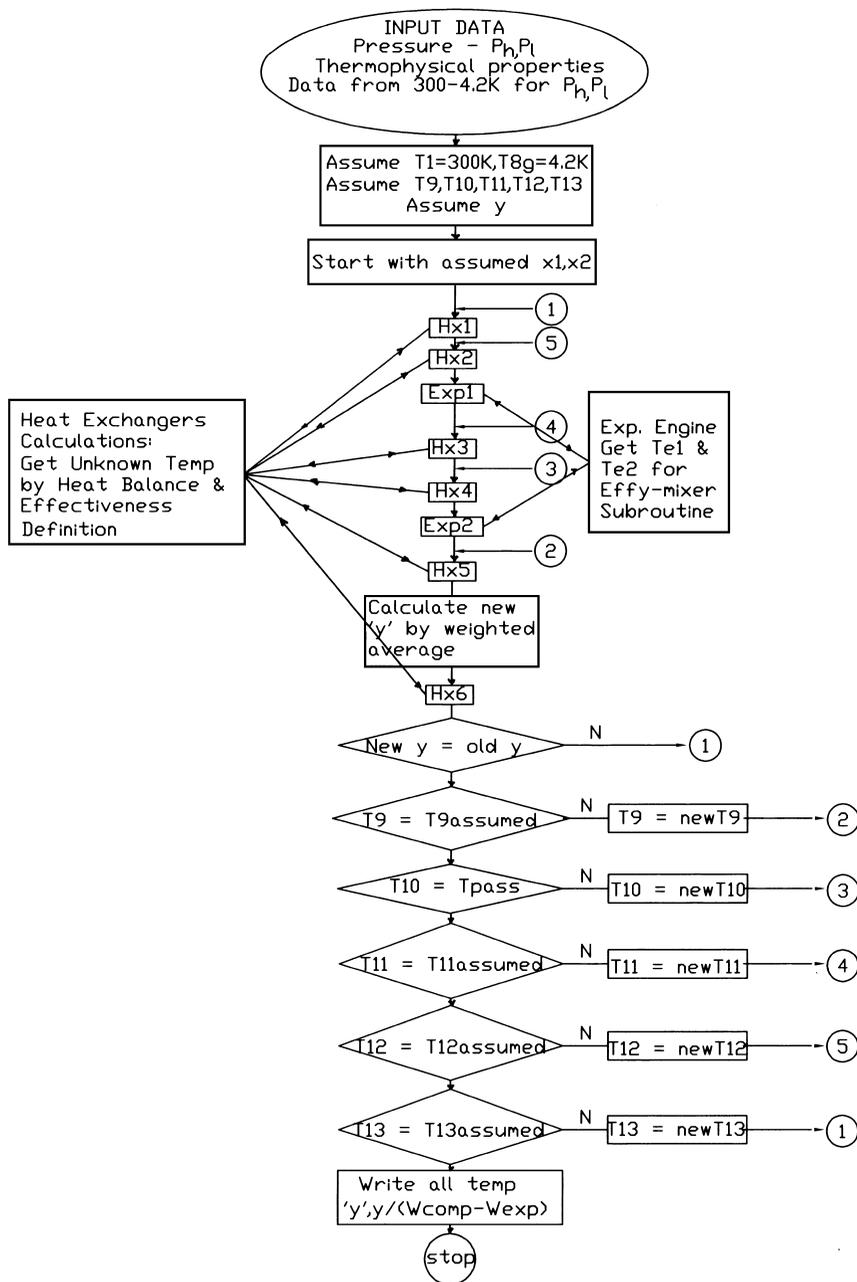


Figure 3 Flow chart for liquefaction cycle analysis

Equation (1) assumes that the liquefaction of helium takes place in all the cases. Let us call the 'y' value obtained from this equation y_1 . However, it is also possible that due to changes in x_1 and x_2 or ϵ and η values, there is no liquefaction of the gas. As a result of this, the isenthalpic line indicating the J-T expansion may not fall in the two-phase region and it may fall outside the dome of the two-phase region. This is taken into account by the bisection equation in the two-phase dome of T-S diagram to ensure liquefaction or no liquefaction cases. Let us call the 'y' value obtained from bisection equation as y_2 which is given below:

$$y_2 = [(h_g - h_7)/h_{fg}]^*(1 - x_1 - x_2) \quad (5)$$

where h_{fg} is the latent heat of evaporation for He at 1 bar.

However, one has to be very careful to use the bisection method alone to determine the value of y . This is due to the fact that it may result in an oscillatory or diverging solutions of the analysis due to very small values of y and therefore 'y' sensitivity of these calculations. As the isenthalpic line may fall in the gas or two-phase region during iterations a very careful approach has to be taken. To overcome this problem, a weighted average method is adopted between the two y values, y_1 value calculated by Equation (1) and y_2 value calculated by Equation (5). Optimum weightage is worked out by various trials of iterations. The optimum combination is determined for two reasons, first to minimise the computer time and second to overcome the oscillating or divergent solutions. The optimum weightage for y_1 and y_2 are found to be 80 and 20%, respectively. Considering this, the resultant y value is given as below:

$$y = (0.8*y_1 + 0.2*y_2) \quad (6)$$

For any liquefier, the y value calculated as y_1 or y_2 should be the same and therefore as criteria for convergence, along with different temperatures, it is ensured that both y_1 and y_2 are practically the same.

To summarise the calculation procedure, the following is the broad outline of different steps of the analysis. The flow chart for the same is given in *Figure 3*.

1. Assume x_1 and x_2 and also the value of y .
2. Assume all the return line temperatures on 1 bar pressure line.
3. Based on the ϵ definitions of the respective heat exchangers followed by enthalpy balances around the heat exchangers, calculate unknown temperatures. As one advances from HX1 towards HX6, correct the earlier assumed temperatures as given in Appendix A.
4. Calculate the temperature of the gas after expansion. Use gas enthalpy mixture formulae to find out resultant temperatures after the mixing of gases from return line after liquefaction and from the expansion engine after expansion. Repeat calculations from HX1 to HX6 with the new temperature values until the same temperatures are obtained.
5. Compute 'y' by Equation (6) and repeat from (3) until y_1 and y_2 are found to be the same in the tolerance limits.

Optimisation of the mass fraction for expanders

It has been found that the mass fractions x_1 and x_2 and also $(x_1 + x_2)$, in case \dot{m} is assumed to be unity, are very

important parameters in order to get liquefaction and also to get maximum yield. The parameters, x_1 , x_2 , $(x_1 + x_2)$, effectiveness of heat exchangers and efficiency of the expanders together determine the liquefier performance. It is obvious that the effectiveness of the heat exchangers and the efficiencies of the expanders should be as high as possible in order to get maximum yield from the liquefier and the higher values are fixed up mostly by fabrication or space limitations. However, x_1 and x_2 , or the sum of x_1 and x_2 are very important parameters in all the types of liquefiers including the ones operated by using turboexpanders. The above analysis is extended to understand the effect of x_1 and x_2 on the output of the liquefier. The parameter OP, to be optimised for a unit total mass flow rate, is given as:

$$OP = \dot{m}_f / (\text{Net Work}) = y / (W_c - \sum x W_e) \quad (7)$$

where W_c is the work done on the compressor and W_e is the work done by the expander per unit mass. A routine is developed to calculate OP parameter for given ϵ set for all the heat exchangers and η of the expanders. The values of x_1 and x_2 are varied during the execution. It is found that the solution of the program diverges for the cases in which no liquefaction occurs and these are considered as limiting cases for the liquefaction.

Results

Optimisation of the mass flow rate through the expanders

It is obvious that the cold produced in the expanders and in the J-T expansion valve is responsible for bringing down the temperature of the helium gas from 300 K to below 7.5 K. The refrigeration effect produced in the expanders is proportional to the mass flow rate directed through them and also to the inlet temperature of the gas of the engine. The refrigeration effect thus produced determines if the machine would function as a liquefier or as a refrigerator depending upon the temperature levels of the expanders. In a similar way, the liquefaction produced in the cycle is directly proportional to the mass flow rate directed to expand through the J-T valve. Considering this, it is really a matter of conflict to decide what fraction of total mass flow rate should be directed through the engines so that the liquefier functions near an optimum value as given above. An optimisation routine is attached to the main program to calculate the OP value for a different fraction of mass flow rates that are directed through the first and the second expansion engines denoted by x_1 and x_2 , respectively. The execution of this routine is quite a computer-intensive task. *Figure 4* gives these results as a plot of OP versus x_1 . The curves are plotted for different values of x_2 .

The aim of this exercise is to find out a combination of x_1 and x_2 for which OP is maximum. In the optimisation routine, x_2 is kept constant and x_1 is varied so as to determine the local maximum OP value, termed $(OP)_{\max}$, for this combination. It is seen that as x_2 decreases from 0.5 to 0.45, the $(OP)_{\max}$ value increases, indicating that $(OP)_{\max}$ obtained by the first combination is not an optimum one. The $(OP)_{\max}$ thus obtained for each x_2 curve, shows an increase up to a certain point only and then starts descending down. The OP value associated with this point indicates an optimum combination of x_1 and x_2 for the present configuration and is termed $(OP)_{\text{opt}}$, which is

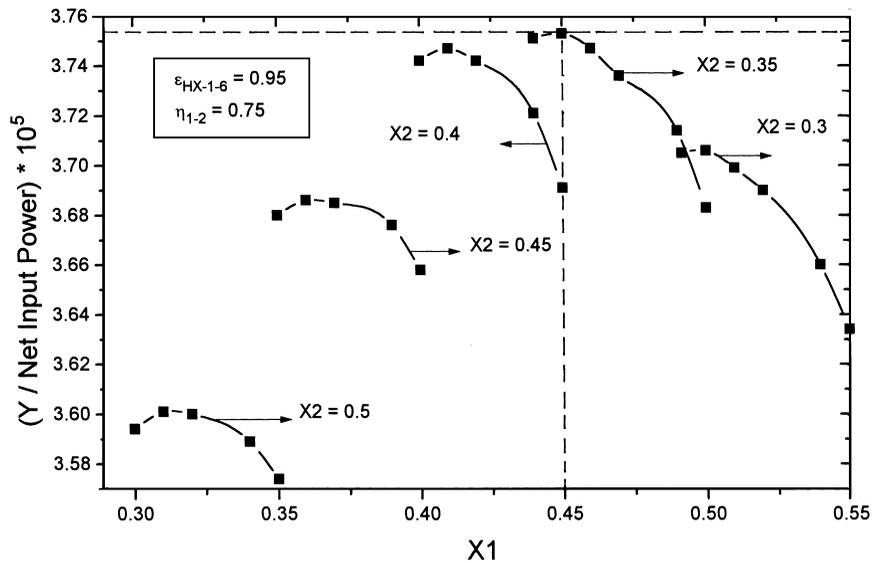


Figure 4 Optimisation of helium mass flow rate fractions through expanders

marked in the figure. It could also be noted that no solutions were obtained for the $(x_1 + x_2)$ combination less than a particular value. This is attributed to the fact that the temperature after the J-T expansion oscillates through liquefier to refrigerator region (inside or outside the two-phase dome) adding an imbalance in the program due to changes in the thermophysical properties of the liquid and gas. To make sure that the machine functions as a liquefier it is safer to conclude that there is a unique value of $(x_1 + x_2)_{\min}$ depending on the operating pressure, ϵ and η combination, below which the machine will not function as a liquefier but as a refrigerator only. The designer should therefore know the relationships of all these parameters before he goes ahead with the design of the heat exchangers and expanders.

It is noticed from the above figure that the combination of $x_1 = 0.45$ and $x_2 = 0.35$ shows the maximum value of $(OP)_{\max}$ as compared to any other combinations of x_1 and x_2 and this is the $(OP)_{\text{opt}}$ value for the given ϵ and η values indicated in the figure. It states that, for this combination of x_1 and x_2 , the output in terms of liquefaction quantity is maximum and the net power input is minimum. The important point to be noted here is that for all the cases, $(OP)_{\max}$ value including $(OP)_{\text{opt}}$ lies at a combination where x_1 and x_2 together constitute about 80–81% of the total mass flow rate while the remaining 19–20% of the total mass flow rate goes through the J-T valve. It is also seen that as the $(x_1 + x_2)$ value is below 79–79.5% there is no liquefaction indicated by the divergence of the program in the present case. This is due to the fact that in these cases, the point of the isenthalpic line after J-T expansion translates into the gaseous region, i.e. outside the dome. As the values of $(x_1 + x_2)$ exceed an optimum value there is a decrease in the OP value essentially due to the fact that effectively less mass flows through the J-T valve and this decreases the values of y in these cases. The results of the present analysis are valid for the liquefiers without LN2 precooling. The ϵ assumed for these cases is 95% for all the heat exchangers and the η assumed for both the expanders is 75%.

To study the effect of increased ϵ of the heat exchangers on this combination of x_1 and x_2 , the routine was executed again. It shows that even if the ϵ of the heat exchangers

increases the optimum point remains almost at the same level, however, the minimum $(x_1 + x_2)$ requirement shifts towards left or on the lesser side. The analysis highlights the fact that for the case of all 98% efficient heat exchangers, the minimum requirement of $(x_1 + x_2)$ is around 75–76% as against 79% for the case of 95% efficient heat exchangers. This explains why in the LN2 pre-cooled liquefiers, which is synonymous with more efficient heat exchangers case, the minimum $(x_1 + x_2)$ values lie around 75% in practice.

Some industrial or actual machine data are available to substantiate the optimum mass flow rate arguments. However, due to the classified nature of the data, these can not be revealed.

Effect of ϵ on temperature distribution and performance of the cycle

It is clear from the T-S diagram that the most important temperature which determines the amount of helium liquefaction is the one before J-T expansion, i.e., T7, and also the mass flow rate through the J-T valve. The purpose of heat exchangers and expanders is mainly to reduce the gas temperature from 300 K to a reasonable value of T7 in order to get liquefaction after the J-T expansion. T7 should necessarily be around 7.5 K maximum for 15 bar pressure to get some liquid yield. If this temperature is lesser than 7.5 K one can expect a higher quantity of liquefaction, however this argument should be critically evaluated looking at the actual T-S diagram. It is quite difficult to bring down this temperature below 6.5 K without increasing the complexity of the cycle, and this can be realised if one has an idea about the feasibility of design of the heat exchangers and the expanders.

It is obvious that as the ϵ of the heat exchanger increases, the performance of the liquefier is better due to the decrease in the final value of T7 for a given mass flow rate through the compressor. However, this does not mean that the temperatures at all the points decrease by the same amount. The fall in T7 could be achieved by various means, i.e. merely by increasing the ϵ of any of the heat exchangers or any two or all the heat exchangers, and could also be due to an increase in the η of any or all the expanders.

Table 1 Temperature distribution for different ϵ of heat exchangers

Sr. no.	ϵ (%)	T2 (K)	T3 (K)	T4 (K)	T5 (K)	T6 (K)	T7 (K)	T9 (K)	T10 (K)	T11 (K)	T12 (K)	T13 (K)	T14 (K)	γ (%)
1	95	232.47	90.18	48.43	20.77	10.46	6.35	9.98	14.54	46.27	73.17	224.5	296.22	5.82
2	96	239.92	90.57	48.11	20.59	10.26	6.27	9.88	14.5	46.37	73.85	233.29	297.33	6.18
3	97	249.27	91.23	47.93	20.44	10.08	6.2	9.79	14.47	46.61	74.72	244.03	298.32	6.52
4	$\epsilon_3 = 97$													
	$\epsilon_{1-6} = 95$	233.52	91.92	47.49	20.61	10.37	6.33	9.9	14.44	46.14	74.8	225.59	296.27	5.93
5	$\epsilon_1 = 97$													
	$\epsilon_{2-6} = 95$	222.18	87.45	46.92	20.51	10.32	6.31	9.85	14.38	44.81	71.01	214.62	297.43	6.01
6	$\epsilon_5 = 97$													
	$\epsilon_{1-6} = 95$	233.11	90.99	18.88	21.12	10.39	6.33	10.10	14.85	46.7	73.85	225.15	296.25	5.91

$\eta_1 = \eta_2 = 75\%$.
 $x_1 = x_2 = 0.4$.

Table 1 gives the values of the temperatures at various locations in the cycle, to understand changes in the temperature distribution across the cycle obtained by the present cycle analysis, when the ϵ of all the heat exchangers or any of the heat exchangers is increased. The expander efficiency is assumed to be 75% for these cases and also that 40% of the total mass flow rate is diverted through each of the expanders. Cases 1, 2 and 3 in the table give the temperature distribution in the liquefier in which the ϵ of all the heat exchangers is changed simultaneously by the same amount. It is seen from the table that as ϵ of all the heat exchangers increases from 95 to 97%, the liquefaction, γ , increases by 12%, which is quite substantial. Also, the temperatures T2 and T3 show an increase with the increase in the effectiveness. However, after the first expansion point the temperature drops down from T4 up to T10 while T11 to T14 shows an increase again. Cases 4, 5 and 6 show the implications of increased ϵ of an individual heat exchanger, keeping ϵ of the rest of the heat exchangers as 95%. The results of these cases could be compared with case 1 where all the heat exchangers have a ϵ of 95%. Cases 4, 5 and 6 show the effect of increased ϵ of 97% for heat exchangers numbers 3, 1 and 5, respectively. It is seen in case 4 that the increase in ϵ decreases the temperatures after HX3 onwards, from T4 to T11. Similarly, case 5 is for the increased ϵ of the first heat exchanger, in which temperatures T2 to T13 show a decreasing trend as compared to

case 1. Case 6 shows the effect for increased ϵ of HX5, wherein only T6 and T7 show a decrease in the temperature. Thus, the table shows that any increase in ϵ of any heat exchanger results ultimately in the decrease in the value of T7, which finally affects the liquefaction process directly.

Figure 5 shows the effect of the ϵ of the heat exchangers on the performance of the liquefier graphically. It shows the effect of variation of the ϵ of a particular heat exchanger on the performance of the liquefier. The figure shows the relative importance of the ϵ of each heat exchanger. It can be seen that the ϵ of heat exchangers 3, 4 and 5 should necessarily be higher in order to ensure liquefaction, while for other heat exchangers, ϵ can have little less values as shown in the curves. The figure also shows that there is a significant change in the performance of the liquefier if the ϵ of all the heat exchangers are increased simultaneously as compared to an increase in individual ϵ of any of the heat exchangers. The curves are significant data to understand the implications of changes in ϵ of any heat exchangers.

Effect of η on the performance of the liquefier

Figure 6 shows the effect of η on the performance of the liquefier. It is quite clear from the curves that as the η increases the performance of the liquefier increases linearly. Also, it shows that if the η_2 is 75%, the minimum η_1 should

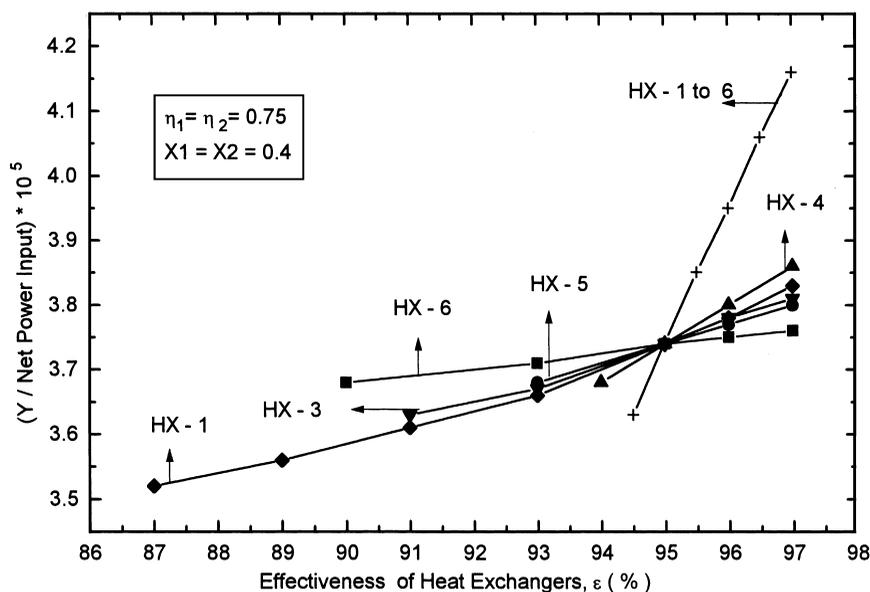


Figure 5 Effect of heat exchanger effectiveness (ϵ) on the performance of the liquefier

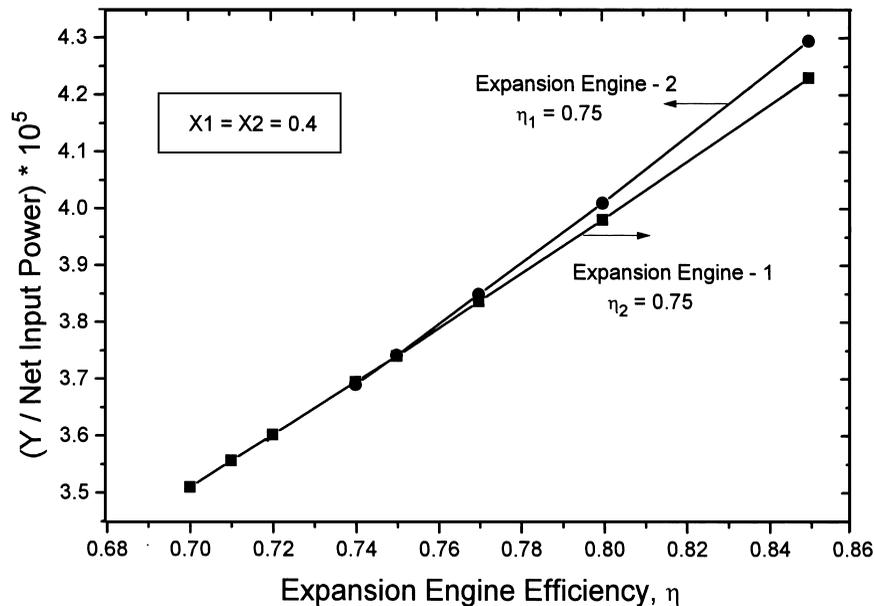


Figure 6 Effect of expander efficiency (η) on the performance of the liquefier

be 70% in order to have liquefaction. Similarly, if η_1 is 75% the minimum η_2 should be 74% in order to have liquefaction. This highlights the importance of minimum η of the expanders and also their interdependence.

Conclusions

The paper presents a cycle simulation for the Collins helium liquefaction cycle with six heat exchangers and two reciprocating expanders. It highlights the concept of an optimum mass flow rate through expanders for the liquefier. At the same time, the paper analytically puts forward the importance of heat exchanger effectiveness (ϵ) and expander efficiency (η) on the performance of the liquefier. The optimum mass flow rate concept holds good for the liquefiers also with the turboexpanders. The simulation can be adapted to bring about any changes in the configuration of the liquefaction cycle and to make a quantitative comparison of different cycles based on their merits and demerits. The analysis is very important to get a preliminary design data for the heat exchangers and the expanders for a required helium liquefaction rate.

References

1. Hubbell, R.H. and Toscano, W.M., Thermodynamic optimisation of helium liquefaction cycles. *Adv. Cryo. Engng.*, 1980, **25**, 551.
2. Minta, M. and Smith, J.L., An entropy flow optimisation technique for helium liquefaction cycles. *Adv. Cryo. Engng.*, 1984, **29**, 469.
3. Khalil, A. and McIntosh, G.E., Thermodynamic optimisation study of the helium multiengine Claude refrigeration cycle. *Adv. Cryo. Engng.*, 1978, **23**, 431.
4. Hilal, M.A., Optimisation of helium refrigerators and liquefiers for large superconducting systems. *Cryogenics*, 1979, **19**, 415.
5. Nobutoki, M., Iwamoto, K. and Matsuda, H., Simulation of the large helium refrigeration plant for LHD. *Proceedings of the 16th ICEC Cryogenics (Suppl)*, Vol. 36. 1996, 71.
6. Malaaen, E., Owren, G., Wadahl A. and Wagner, U., Simulation program for cryogenic plants at CERN. *Proceedings of the 16th ICEC Cryogenics (Suppl)*, Vol. 36. 1996, 99.
7. Van Sciver, S.W., Helium cryogenics. Plenum Press, New York, USA, 1986.

Appendix A

The thermal analysis of the liquefier involves solution of following equations in an iterative manner. *Figures 1 and 2* should be referred to to understand the nodal nomenclature used in these equations. Excepting temperatures at points 2 and 8g from these figures, no other temperatures are known. Temperatures $T13_d$, $T12_d$, $T11_d$, $T10_d$ and $T9_d$ indicate assumed values for $T13$, $T12$, $T11$, $T10$ and $T9$, respectively. The temperature values which could be obtained from these equations are indicated at the right side of the arrow. The suffix i in the following equations indicates the temperature of the gas under ideal conditions of heat exchange.

Heat exchangers 1, 2 and 4

Heat exchangers 1, 2 and 4 are the cases where the capacity rate on the warm side is higher than that on the cold side. Therefore, the equations for computing heat balance are similar in nature. In the case of heat exchanger 1, $T14_i$ could be equal to $T1$. So the enthalpy of gas at pressure P_i and temperature $T14_i$ can be given as:

$$h14_i = h(T1, P_i) \quad (\text{A.1})$$

Applying heat exchanger effectiveness definition:

$$h14 = \epsilon_1(h14_i - h13_d) + h13_d = > T14 \quad (\text{A.2})$$

Applying enthalpy balance:

$$h2 = h1 - (1 - y)(h14 - h13_d) = > T2 \quad (\text{A.3})$$

The equations for heat exchangers 2 and 4 should be similar to the ones given above. However, it is always a good practice to verify the capacity rates of each stream in each case due to the fact that C_p of helium goes on increasing at lower temperatures.

Heat exchangers 3 and 5

For heat exchangers 3 and 5, the capacity rate on the cold side will be more than the warm side. This changes the enthalpy balance relationship as compared to heat exchangers 1, 2 and 4. For heat exchanger 3, T_{4_i} could be equal to T_{11} . So, the enthalpy of gas at pressure P_h and temperature T_{11_d} can be given as:

$$h_{4_i} = h(T_{11_d}, P_h) \quad (\text{A.4})$$

$$h_4 = h_3 - \epsilon_3(h_3 - h_{4_i}) = > T_4 \quad (\text{A.5})$$

$$h_{12} = h_{11_d} + (1 - x_1)(h_3 - h_4)/(1 - y_1) \quad (\text{A.6})$$

$$= > T_{12}$$

Mixer 1 and 2

After expansion, the expanded gas mixes with the return stream coming back after the liquefaction. The resultant temperature of the stream after mixing depends on temperature and respective mass flow rates of the two streams. In the case of expander 1, he_{1_d} is the enthalpy of the gas at a point just after isentropic expansion from temperature T_3 and he_1 is the enthalpy of the gas after expansion taking into consideration the isentropic efficiency, η_1 , defined in Equation (4). $T_{11_{id}}$ is the temperature of the return stream

gas at low pressure after the heat exchanger 4 before mixing with expansion stream at temperature Te_1 . From the definition of η_1 the following equation could be obtained:

$$he_1 = h_3 - \eta_1(h_3 - he_{1_d}) = > Te_1 \quad (\text{A.7})$$

Mixer equations for three gas streams:

$$h_{11_{id}} = [(1 - y)(h_{11_d}) - (x_1 * he_1)] / (1 - x_1 - y) \quad (\text{A.8})$$

$$= > T_{11_{id}}$$

Mixer equation for three gas streams at 11:

$$h_{11} = [(1 - x_1 - y)(h_{11_i}) + (x_1 * he_1)] / (1 - y) \quad (\text{A.9})$$

$$= > T_{11}$$

In a similar way, the equations for mixer 2 are established.

Heat exchanger 6

The analysis of this heat exchanger has to be correctly carried out as the inlet temperature on the cold side is very near to boiling point of He and C_p of the gas at this temperature is quite high. So, one has to verify in what category this heat exchanger falls, and accordingly it has to be evaluated as given above.